

On the Correspondence of Time-Delay and Spatially Extended Systems

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Z. Naturforsch. **52 a**, 573–577 (1997); received June 10, 1997

We establish a straightforward connection between spatially extended systems, the dynamics of which are modeled with the help of partial differential equations and time-delay systems. To this end, we give a linear partial differential equation with a nonlinear boundary condition whose solutions are equivalent to the solutions of a time-delay differential equation. We observe that the phase space of these systems exhibits a pronounced structure. In this paper, we express the structure of the phase space of time-delay systems and the corresponding spatially extended system by distinguishing between a ‘linear subsystem’ and a ‘localized nonlinearity’. We find that the high-dimensional chaotic dynamics observed in time-delay systems is fundamentally different from the spatio-temporal chaos observed in homogeneous spatially extended systems, the dynamics of which is modeled with the help of *nonlinear* partial differential equations. To this end, we investigate the space-time correlation function and the ‘thermodynamic limit’.

Time-delay systems have attracted considerable interest in the field of nonlinear dynamics because of their ability to exhibit high-dimensional chaotic motion [1] characterized by an increasing number of positive Lyapunov exponents with increasing delay time. Therefore, time-delay systems, together with nonlinear partial differential equations [2] and suitably chosen ordinary differential equations [3]–[6] are major examples for hyperchaos [7] and the chaotic hierarchy [8]. Recently, it has been shown that a time-delay system can be mapped on a spatial one (with a discrete time), which then produces spatio-temporal chaos with characteristic scaling laws of the Lyapunov exponents [9]–[11]. The purpose of this paper is to give a linear partial differential equation with a nonlinear boundary condition, the solutions of which are identical to those of a time-delay system. This leads us directly to the structure of the phase space of time-delay systems, which will be expressed by distinguishing between a ‘linear subsystem’ and a ‘localized nonlinearity’. Finally, we discuss the space-time correlation function and the ‘thermodynamic limit’ of both systems.

Consider the N -dimensional first-order wave equation

$$\partial_t \mathbf{u}(x, t) - v \partial_x \mathbf{u}(x, t) = 0, \quad (1)$$

where $\mathbf{u}(x, t)$ is an N -dimensional vector field in one spatial dimension $x \in [0, L]$ and v is the velocity. Let $\mathbf{u}_0(t) := \mathbf{u}(0, t)$ and $\mathbf{u}_L(t) := \mathbf{u}(L, t)$ be the values of $\mathbf{u}(x, t)$ at the boundaries. We take the boundary condition as

$$\dot{\mathbf{u}}_L(t) = \mathbf{h}(\mathbf{u}_L(t), \mathbf{u}_0(t)). \quad (2)$$

The initial condition is given by N functions

$$\mathbf{u}(x, 0) = \mathbf{u}_i(x), \quad 0 \leq x \leq L \quad (3)$$

on an interval of length L . The phase space of system (1) is \mathcal{C}_L^N , the space of the N -dimensional differentiable functions defined on the interval $[0, L]$. The dimension of the phase space is infinite. In Fig. 1, we show a typical time evolution of a scalar partial differential equation (1) with a Mackey-Glass type nonlinear boundary condition of the form (2).

The partial differential equation (1) is linear, but the boundary condition (2) is nonlinear and introduces a nonlocal coupling of variables in space. The first-order wave equation (1) only allows the unidirectional, dispersionless transport of signals and, therefore, can be integrated,

$$\mathbf{u}(x, t) = \mathbf{f}(x + vt), \quad (4)$$

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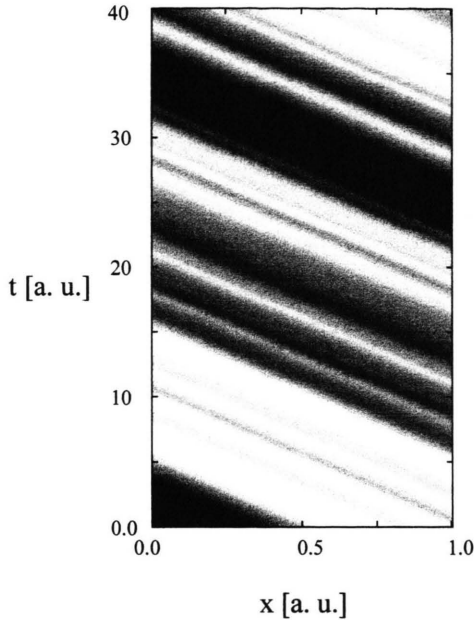


Fig. 1. Time evolution of the scalar partial differential equation $\partial_t u(x, t) - 0.1 \partial_x u(x, t) = 0$, $x \in [0, 1]$ with the nonlinear boundary condition $\dot{u}_1(t) = \frac{3.0 u_0(t)}{1 + u_0(t)^{10}} - u_1(t)$. The values of $u(x, t)$ are transformed to a grey scale and shown for successive times.

with an arbitrary, differentiable N -dimensional function f . Hence, there is a functional relationship between the values of \mathbf{u} at different locations, $x_1 < x_2$,

$$\begin{aligned} \mathbf{u}(x_1, t) &= f(x_1 + vt) \\ &= f(x_2 + v(t - \frac{x_2 - x_1}{v})) \\ &= \mathbf{u}(x_2, t - \tau_{1,2}), \end{aligned} \quad (5)$$

with $\tau_{1,2} = (x_2 - x_1)/v$. In the case of the scalar spatially extended system, the dynamics of which is shown in Fig. 1, the uni-directional, dispersionless transport of signals causes the observed stripe-like structures. Specifically, there is a functional relationship between the values of \mathbf{u} at the boundaries,

$$\mathbf{u}_0(t) = \mathbf{u}_L(t - \tau_0), \quad (6)$$

with $\tau_0 = L/v$.

Inserting the functional relationship (6) into the nonlinear boundary condition (2) gives the result

$$\dot{\mathbf{u}}_L(t) = \mathbf{h}(\mathbf{u}_L(t), \mathbf{u}_L(t - \tau_0)), \quad (7)$$

which is an N -dimensional time-delay differential equation for the variables $\mathbf{u}_L(t)$. Therefore, the solutions of the N -dimensional linear partial differential equation (1) with the nonlinear boundary condition (2) correspond to the solutions of an N -dimensional nonlinear time-delay differential equation (7). The nonlocal couplings in space of the partial differential equation are equivalent to the nonlocal couplings in time, i. e., the memory, of the time-delay equation (7).

As mentioned above, the phase space of time-delay systems as well as the phase space of the partial differential equation (1) is infinite dimensional, but the phase space exhibits a characteristic structure. In this paper, the structure is expressed by dividing the phase space into two different subsystems. We feel that the two subsystems reveal their nature most clearly if we consider the couplings of the infinite number of variables of the partial differential equation (1). We distinguish between two cases:

- The variables $\mathbf{u}(x, t)$, with $0 \leq x < L$, solely couple linearly to their neighboring variables via (1). This leads us to call this infinite number of variables ‘linear subsystem’. The linear coupling of the linear subsystem only allows for the uni-directional and dispersionless transport of signals, which is expressed as a functional relationship between the variables $\mathbf{u}(x, t)$ with $0 \leq x < L$, according to (5). The linear subsystem is not able to initiate any unstable behavior of the system.

- The variables $\mathbf{u}_L(t)$ are able to couple nonlinearly to themselves and the variables $\mathbf{u}_0(t)$ via the function \mathbf{h} of the boundary condition (2). If the dynamics of the partial differential equation (1)-(2) is chaotic, the chaos-generating mechanism of nonlinear stretching and folding is brought into the dynamics by this nonlinear coupling of the variables $\mathbf{u}_L(t)$. In this sense, the chaos-generating nonlinearity is ‘located’ in the variables $\mathbf{u}_L(t)$ and, consequently, we call these variables ‘localized nonlinearities’.

In Fig. 2, we give an illustration of the division of phase space into a linear subsystem and a localized nonlinearity. A snapshot of the time evolution of a scalar system $u(x, t)$ of the form (1) is shown.

We have confirmed that there is a functional relationship among the variables in phase space via (5) as a direct consequence, or manifestation, of the structure of phase space. More specifically, the variables of the linear subsystem are functionally related to the variables of the localized nonlinearity,

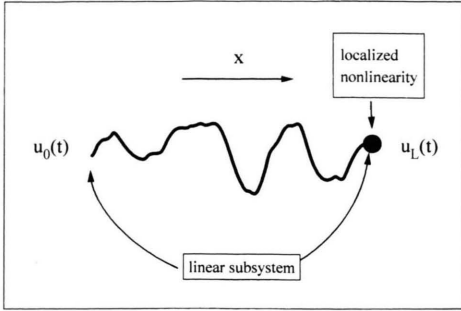


Fig. 2. Illustration of the division of the system into a linear subsystem and a localized nonlinearity. A state $u(x, t')$ of a scalar system (1) is shown for a fixed time t' .

$$u(x, t) = u_L(t - \tau), \quad 0 \leq x < L, \quad (8)$$

with $\tau = (L - x)/v$. Therefore, if the velocity v and the length L are known or, equivalently, in the case of time-delay systems, the delay time $\tau_0 = L/v$ is known, the time evolution of all variables in phase space can be exactly determined by only observing the dynamics of the localized nonlinearity. In [12] - [16], we have proposed a method for time series analysis to uniquely identify the delay time of time-delay systems from the time series. With this it is possible to construct the trajectory in the infinite dimensional phase space of an *unknown* time-delay system only by observing the time evolution of the localized nonlinearity. In contrast to the well-known embedding techniques for the reconstruction of the trajectory in phase space [17], the construction of the trajectory of a time-delay system in phase space via (8) is not restricted to the dynamics on low-dimensional attractors, but also applies to transient motion as well as arbitrary high-dimensional chaotic motion.

The above arguments indicate that the structure of the phase space of time-delay systems as well as of linear partial differential equations with a nonlinear boundary condition is fundamentally different from the structure of the phase space of ordinary differential equations and nonlinear partial differential equations. In the case of nonlinear partial differential equations, every location x is coupled nonlinearly with its neighbors, at least via one variable, and is able, in principle, to generate instabilities. Also, in the case of ordinary differential equations, despite of the fact that the couplings can be chosen specifically so that the system can also be divided into a linear subsystem and a localized nonlinearity [3] - [6], the nonlinear interactions

can be located, in principle, in all variables in phase space. On the other hand, we want to emphasize that the division of the phase space of an infinite dimensional dynamical system into a linear subsystem and a localized nonlinearity in the sense given above does not only apply to time-delay systems or equivalently to spatially extended systems with a time-evolution equation (1) - (2). It also applies to linear partial differential equations (not only of the form (1)) with a nonlinear boundary condition (2). In this case, the transfer function takes a more complicated form than a simple time delay as in (6). This leads to a functional differential equation [18]. With some modifications, the division of the phase space into a linear subsystem and a localized nonlinearity can also be accomplished in this case. This will be shown elsewhere.

In this paragraph, we analyze the normalized correlation function with respect to space and time

$$C(\chi, \tau) = \quad (9)$$

$$\frac{\langle\langle u(x - \chi, t - \tau), u(x, t) \rangle\rangle - \langle\langle u(x, t) \rangle\rangle^2}{\langle\langle u(x, t)^2 \rangle\rangle - \langle\langle u(x, t) \rangle\rangle^2}$$

of the system (1), where $\langle\langle \dots \rangle\rangle$ denotes the average over space and time. Applying (5), we find

$$C(\chi, \tau) = C(\chi - vT, \tau + T) \quad (10)$$

for any time T as a direct consequence of the unidirectional, dispersionless transport of signals. If we choose $T = -\tau$, (10) reduces to

$$C(0, \tau) = C(\chi, 0) \quad (11)$$

with $\chi = v\tau$. Equation (11) indicates the equality of the space-averaged correlation function with respect to time and the time-averaged correlation function with respect to space. Therefore, if system (1) has a finite correlation time τ_c due to an unstable, chaotic dynamics, the correlation length χ_c is given by

$$\chi_c = v\tau_c \quad (12)$$

and is, therefore, also finite. The system (1) seems to exhibit spatio-temporal chaos according to the criteria given in the literature (see, e. g., [2]). Nevertheless, it is the opinion of the authors that the high-dimensional chaos observed in the system (1) is purely temporal and not spatio-temporal, because (10) clearly shows

that the correlations do not decay in a frame of reference which moves with the velocity $-v$. Therefore, the correlation time in the moving frame of reference equals the time a signal needs to traverse the system, $\tau_{c,v} = vL$. This result does not contradict to results published recently by Giacomelli and co-workers [9]–[11], who attributed the high-dimensional chaotic motion of time-delay systems to spatio-temporal chaos. In the latter work, the identification of space as well as time differs from that in this paper.

In this paragraph, we would like to discuss the large-volume limit, $L \rightarrow \infty$, of the spatially extended systems (1) or, equivalently, the limit of an increasing delay time, $\tau_0 \rightarrow \infty$, of the corresponding time-delay system (7). We relate the results to the ‘thermodynamic limit’, which has been discussed in the literature. Generally, it is believed that the attractor dimension of a time-delay system is proportional to the delay time, $D \propto \tau_0$. This has been confirmed for several time-delay models with the help of a numerical estimation of the fractal dimension D [1, 19, 20]. For the dynamics of the partial differential equation (1), this means that the attractor dimension is an extensive quantity $D \propto L$. On the other hand, it has been confirmed that the metric entropy of time-delay systems approaches a constant value $h \rightarrow h_0$ for $\tau_0 \rightarrow \infty$, which must equally hold for the system (1) for $L \rightarrow \infty$. Therefore, the proportionality of the metric entropy with the volume, $h \propto V$, and the characteristic scaling of the Lyapunov exponents

$$\lambda_i \approx f_i(i/L), \quad (13)$$

as conjectured by Grassberger [21] and observed by Livi *et al.* [22] in the case of homogeneous spatially extended systems, cannot hold for the system (1) and (7), even though a scaling of *comoving Lyapunov exponents* has been observed [11]. Thus, the large-volume limit of the inhomogeneous system (1) is fundamentally different from the large-volume limit of homogeneous systems. Here, we suggest an explanation of this observation with the help of the concept of localized nonlinearities, as introduced in one of the preceding paragraphs. The linear subsystem $\mathbf{u}(x, t)$, with $0 \leq x < L$, only allows the dispersionless transport of signals and the information remains unchanged. Therefore, the dynamics of the linear subsystem cannot be responsible for the positive metric entropy. It is the localized nonlinearity $\mathbf{u}_L(t)$ which allows for the nonlinear processing of information

and, consequently, the localized nonlinearities must be considered as the source of the observed positive metric entropy. The ‘flow of information’ passing the localized nonlinearities is proportional to the correlation time τ_c . It is independent of L or τ_0 . Therefore, the metric entropy is approximately constant in the large-volume limit, in agreement with the observations. This argument leads us to conjecture that the metric entropy is proportional to the number of localized nonlinearities N_{NL} divided by the correlation time τ_c ,

$$h \propto \frac{N_{NL}}{\tau_c}. \quad (14)$$

To our knowledge, there is no other chaotic indicator which scales like the above equation. Therefore, it might be possible to estimate the number of localized nonlinearities of an *unknown* dynamical system with the help of (14). In the case of homogeneous, nonlinear spatially extended systems, it is expected that the number of localized nonlinearities scales linearly with the volume, $N_{NL} \propto V$, recovering the proportionality of the metric entropy with the volume, as proposed by Grassberger [21] and Livi *et al.* [22]. As a direct consequence of (14), we expect the metric entropy to be proportional to the attractor dimension, $h \propto D$, in the limit $\tau_c \rightarrow 0$ with a constant value of τ_0 . This limit corresponds to a rescaling of the time and is in accordance with the results obtained by Lepri and coworkers [23] in the case of delayed maps. The question whether in the latter limit the Lyapunov exponents show a scaling function, see (13), as has been observed in the case of spatio-temporal chaos, and in the case of the comoving Lyapunov exponents of time-delay systems [11] remains an important open question.

In conclusion, we have given a linear partial differential equation with a nonlinear boundary condition, whose solution is equivalent to the solution of a time-delay differential equation indicating that there is a strict correspondence between spatial systems and time-delay systems. Additionally, the clear difference between spatially extended systems, the dynamics of which is determined by a *nonlinear* partial differential equation and time-delay systems, is uncovered by analyzing the couplings in phase space. We find that the infinite dimensional phase space of time-delay systems, as well as the corresponding linear partial differential equation, can be divided into

a chaos-generating, finite dimensional localized non-linearity and an infinite dimensional linear subsystem. Additionally, we have studied the space-time correlation function, which reveals that the high-dimensional chaotic dynamics observed in the case of time-delay differential equations as well as of the corresponding linear partial differential equation is not similar to spatio-temporal chaos. This conjecture is further strengthened by considering the large-volume limit of the spatial system, or equivalently, the large-delay-time limit of the corresponding time-delay system. In both cases, the attractor dimension grows proportional to the volume (delay time). The metric

entropy, though, approaches a constant value for increasing volume (delay time). We conjecture that the large-volume (large-delay-time) limit is directly connected to the number of localized nonlinearities of the system.

Acknowledgements

We thankfully acknowledge valuable discussions with R. Friedrich, H. Kantz, J. Peinke, A. Politi, G. Giacomelli, S. Lepri, O. E. Rössler, and the ENGADYN group. Financial support of the Deutsche Forschungsgemeinschaft is acknowledged.

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